Math. Nachr. **286**, No. 11–12, 1191 – 1207 (2013) / **DOI** 10.1002/mana.201200095

Scattering for strictly hyperbolic systems with time-dependent coefficients

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Received 4 April 2012, revised 26 December 2012, accepted 22 January 2013
Published online 1 April 2013

Key words Hyperbolic systems, time-dependent coefficients, scattering theory
MSC (2010) 35L05, 35L10

The present paper is devoted to finding conditions on the occurrence of scattering for strictly hyperbolic systems with time-dependent coefficients whose time-derivatives are in L^1 in time. More precisely, it will be shown that the solutions are asymptotically free if the coefficients are stable in the sense that their improper Riemann integrals converge as $t \to \pm\infty$, while each nontrivial solution with radially symmetric data is never asymptotically free provided that the coefficients are not stable as $t \to \pm\infty$. As a by-product, wave and scattering operators can be constructed. An important feature is that assumptions on only one derivative of the coefficients are made so that the results would be applicable to the asymptotic behaviour of Kirchhoff systems.

1 Introduction

In this paper we investigate scattering for the Cauchy problem of strictly hyperbolic systems with time-dependent coefficients. The important feature for these results is that we limit the conditions on taking only one derivative of the coefficients keeping it in the framework of the study of asymptotic behaviours for Kirchhoff systems. In [5] the first author gave a sufficient condition on the existence of scattering states for wave equations, and found a special data for non-existence of scattering states. More precisely, there exists a solution $u = u(t, x)$ of the Cauchy problem to strictly hyperbolic equation of second order of the form

$$\partial_t^2 u - c(t)^2 \Delta u = 0$$

such that u is not asymptotically free, where we assume that $c(t) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$ satisfies

$$\inf_{t\in\mathbb{R}} c(t) > 0, \quad c'(t) \in L^1(\mathbb{R}), \quad \lim_{t\to\pm\infty} c(t) = c_{\pm\infty} > 0,$$

and the improper Riemann integrals of $c(t)-c_{\pm\infty}$ do not exist. On the contrary, if the improper Riemann integrals of $c(t) - c_{\pm\infty}$ exist, then each solution u is asymptotically free. As to the strictly hyperbolic equations of second order for "bounded domains," a similar result was obtained in [1]. It should be noted that the results of [5] are applied to deduce non-existence of scattering states for the Kirchhoff equation (see [6]). In this sense the behaviour of $c(t) - c_{\pm\infty}$ affects the development of scattering theory for wave equations with time-dependent coefficients as well as for the Kirchhoff equation.

The first order systems often appear in the analysis of equations of orders larger than two, and of coupled equations of second order (see Examples 1.4–1.5 below). In the present paper we will find conditions on the

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occurrence of scattering for strictly hyperbolic systems with time-dependent coefficients, and as a result, these observations will provide some generalisations of the results of [5]. We will also construct wave operators and scattering operators by using an asymptotic integration method, which was developed in [8], so that such an approach would yield an improvement for the second order equations as well. In hyperbolic systems it is natural to impose a stability condition on the characteristic roots of the symbol of the differential operator.

Apart from the scattering problem, dispersion and Strichartz-type estimates for hyperbolic systems are also of great interest. Large hyperbolic systems appear in many applications, for example the Grad systems of gas dynamics, hyperbolic systems in the Hermite-Grad decomposition of the Fokker-Planck equation, etc. Thus, for general hyperbolic equations with constant coefficients a comprehensive analysis of dispersive and Strichartz estimates was carried out in [14]. The dispersion for scalar equations based on the asymptotic integration method was analysed by the authors in [8], motivated by the higher order Kirchhoff equations. Decay rates of solutions for time-dependent hyperbolic systems without L^1-condition on time-derivatives have been obtained in [15], but in this case one has to make assumptions on a larger number of derivatives of coefficients, making it non-applicable to the theory of Kirchhoff equations. Dispersive estimates in these settings are based on the multi-dimensional version of the van der Corput lemma established in [12, 13]. Optimal dispersion and Strichartz estimates for hyperbolic systems with time-dependent coefficients will be discussed in [9, 10] and will appear elsewhere, as well as the applications to Kirchhoff systems of the results obtained there and in the present paper.

Let us consider the Cauchy problem

$$D_t U = A(t, D_x)U \quad \text{with} \quad D_t = -i\partial_t \quad \text{and} \quad D_{x_j} = -i\partial_{x_j} \ (j = 1, \dots, n), \tag{1.1}$$

$i = \sqrt{-1}$, for $t \neq 0$, with Cauchy data

$$U(0, x) = {}^T(f_0(x), \dots, f_{m-1}(x)) \in \left(L^2(\mathbb{R}^n)\right)^m. \tag{1.2}$$

The operator $A(t, D_x)$ is a first order $m \times m$ pseudo-differential[1] system, namely, its symbol $A(t, \xi)$ is assumed to be of the form

$$A(t, \xi) = (a_{jk}(t, \xi))_{j,k=1}^m,$$

where functions $a_{jk}(t, \xi)$ are positively homogeneous of order one in ξ, $a_{jk}(t, \lambda\xi) = \lambda a_{jk}(t, \xi)$ for $\lambda > 0$, $\xi \in \mathbb{R}^n\backslash 0$, and satisfy

$$a_{jk}(t, \xi/|\xi|) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}; L^\infty(\mathbb{R}^n\backslash 0)) \quad \text{and} \quad \partial_t a_{jk}(t, \xi/|\xi|) \in L^1(\mathbb{R}; L^\infty(\mathbb{R}^n\backslash 0)) \tag{1.3}$$

for $j, k = 1, \dots, m$. We will also assume that $D_t - A(t, D_x)$ is a strictly hyperbolic operator:

$$\det(\tau I - A(t, \xi)) = 0 \text{ has (in } \tau\text{) real and distinct roots } \varphi_1(t, \xi), \dots, \varphi_m(t, \xi) \tag{1.4}$$

for all $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^n\backslash 0$, i.e.,

$$\inf_{t\in\mathbb{R}, |\xi|=1} |\varphi_j(t, \xi) - \varphi_k(t, \xi)| \geq d > 0 \quad \text{for} \quad j \neq k. \tag{1.5}$$

Notice that each characteristic root $\varphi_j(t, \xi)$ is positively homogeneous of order one in ξ. If we do not care about the asymptotic behaviour of amplitudes, it is enough to assume that $a_{jk}(t, \xi/|\xi|)$ are of bounded variation in t. However, the assumption (1.3) on Lipschitz continuity of $a_{jk}(t, \xi/|\xi|)$ in t assures the existence of limiting functions. Namely, there exist $a^\pm_{jk}(\xi)$, $j, k = 1, \dots, m$, such that

$$a_{jk}(t, \xi/|\xi|) \to a^\pm_{jk}(\xi/|\xi|) \text{ uniformly in } \xi \in \mathbb{R}^n\backslash 0 \text{ as } t \to \pm\infty. \tag{1.6}$$

Indeed, the first assumption in (1.3) implies that we have

$$a_{jk}(t, \xi/|\xi|) = a_{jk}(s, \xi/|\xi|) + \int_s^t \partial_\tau a_{jk}(\tau, \xi/|\xi|)\, d\tau,$$

[1] We note that it is important to allow $A(t, D_x)$ to be pseudo-differential (or, rather, it is a time-dependent Fourier multiplier) here since we want the results to include scalar higher order equations as well, e.g. see Example 1.4.

and we conclude from the L^1-assumption on $\partial_t a_{jk}(t, \xi/|\xi|)$ that there exist the limiting functions $a^{\pm}_{jk}(\xi/|\xi|)$ such that

$$a^{\pm}_{jk}(\xi/|\xi|) = a_{jk}(0, \xi/|\xi|) + \int_0^{\pm\infty} \partial_\tau a_{jk}(\tau, \xi/|\xi|)\, d\tau,$$

which satisfy (1.6). Thus, a question naturally arises, whether or not the solution $U(t, x)$ of (1.1)–(1.2) is asymptotic to some solution of the following hyperbolic systems with constant coefficients as $t \to \pm\infty$:

$$D_t V = A_{\pm}(D_x)V. \tag{1.7}$$

Here $A_{\pm}(D_x)$ is an $m \times m$ first order pseudo-differential system, with symbol

$$A_{\pm}(\xi) = \big(a^{\pm}_{jk}(\xi)\big)_{j,k=1}^{m}.$$

Since the characteristic roots depend continuously on the coefficients, it follows from (1.3)–(1.6) that the operators $D_t - A_{\pm}(D_x)$ are strictly hyperbolic. Indeed, it will be shown in Proposition 2.1 that there exist the limiting phases $\varphi^{\pm}_j(\xi)$ of $\varphi_j(t, \xi)$ for $j = 1, \dots, m$:

$$\lim_{t \to \pm\infty} \varphi_j(t, \xi) = \varphi^{\pm}_j(\xi) \text{ uniformly on the sphere } |\xi| = 1. \tag{1.8}$$

Hence, by using (1.5), we have also

$$\inf_{|\xi|=1} |\varphi^{\pm}_j(\xi) - \varphi^{\pm}_k(\xi)| \geq d > 0 \quad \text{for} \quad j \neq k. \tag{1.9}$$

We are now in a position to state our results. For this purpose, let us recall the notion of scattering states. We say that a solution $U(t, x)$ of

$$D_t U = A(t, D_x)U$$

is asymptotically free in $\big(L^2(\mathbb{R}^n)\big)^m$, if it is asymptotic to some solutions $V_{\pm}(t, x)$ of

$$D_t V = A_{\pm}(D_x)V$$

such that

$$\|U(t, \cdot) - V_{\pm}(t, \cdot)\|_{(L^2(\mathbb{R}^n))^m} \longrightarrow 0 \quad (t \longrightarrow \pm\infty).$$

We shall prove here the following theorem.

Theorem 1.1 *Assume* (1.3)–(1.5). *Then the following assertions hold*:

(i) *If the improper Riemann integrals*

$$\Theta^{\pm}_j(\xi) := \int_0^{\pm\infty} \big(\varphi_j(s, \xi) - \varphi^{\pm}_j(\xi)\big)\, ds, \quad j = 1, \dots, m, \tag{1.10}$$

exist for each $\xi \in \mathbb{R}^n \backslash 0$, then each solution $U(t, x) \in C\big(\mathbb{R}; \big(L^2(\mathbb{R}^n)\big)^m\big)$ of (1.1)–(1.2) *is asymptotically free in $\big(L^2(\mathbb{R}^n)\big)^m$. Moreover, the mappings* (*the inverse of the wave operators $\mathscr{W}_{\pm}$*)

$$\mathscr{W}_{\pm}^{-1} : U(0, \cdot) \longmapsto V_{\pm}(0, \cdot)$$

are well-defined and bounded on $\big(L^2(\mathbb{R}^n)\big)^m$.

(ii) *Assume that the initial data $U(0, x)$ are radially symmetric. If the improper Riemann integrals* (1.10) *of $\varphi_j(s, \xi) - \varphi^{\pm}_j(\xi)$ diverge for a.e. $\xi \in \mathbb{R}^n \backslash 0$, i.e., $\big|\Theta^{\pm}_j(\xi)\big| = +\infty$, then non-trivial solutions $U(t, x) \in C\big(\mathbb{R}; \big(L^2(\mathbb{R}^n)\big)^m\big)$ of* (1.1)–(1.2) *are never asymptotically free in $\big(L^2(\mathbb{R}^n)\big)^m$.*

Microlocalising, we also have a version of these statements in cones in the frequency space.

There are different sufficient criteria for (1.10) to hold. For example, if $A(t,\xi/|\xi|) \in C^1\left(\mathbb{R}; \left(L^\infty(\mathbb{R}^n\backslash 0)\right)^{m^2}\right)$ and we assume that

$$t\left(\varphi_j(t,\xi) - \varphi_j^\pm(\xi)\right) = o(1) \text{ uniformly on } |\xi| = 1 \text{ as } t \to \pm\infty, \tag{1.11}$$

and

$$t\partial_t \varphi_j(t,\xi/|\xi|) \in L^1(\mathbb{R}; L^\infty(\mathbb{R}^n\backslash 0)) \quad \text{for} \quad j = 1,\dots,m, \tag{1.12}$$

then (1.10) follows. Indeed, in this case we have $\varphi_j(t,\xi/|\xi|) \in C^1(\mathbb{R}; L^\infty(\mathbb{R}^n\backslash 0))$ for all j, and the statement follows from the trivial identity

$$\int_0^t s\partial_s \varphi_j(s,\xi)\, ds = \int_0^t \left(\varphi_j^\pm(\xi) - \varphi_j(s,\xi)\right) ds + t\big(\varphi_j(t,\xi) - \varphi_j^\pm(\xi)\big).$$

Thus, condition (1.10) follows from (1.11)–(1.12).

Next we state a result on the existence of wave operators. Let us consider the Cauchy problems for strictly hyperbolic systems with constant coefficients

$$D_t V_\pm = A_\pm(D_x)V_\pm, \quad x \in \mathbb{R}^n, \quad \pm t > 0, \tag{1.13}$$

with Cauchy data

$$V_\pm(0,x) = {}^T\left(f_0^\pm(x),\dots,f_{m-1}^\pm(x)\right), \tag{1.14}$$

where $A_\pm(D_x)$ are pseudo-differential operators with symbols $(a_{jk}^\pm(\xi))_{j,k=1}^m$. We will assume that the characteristic roots $\varphi_1^\pm(\xi),\dots,\varphi_m^\pm(\xi)$ of the operators $D_t - A_\pm(D_x)$ are real and distinct, i.e.,

$$\det(\tau I - A_\pm(\xi)) = \left(\tau - \varphi_1^\pm(\xi)\right)\cdots\left(\tau - \varphi_m^\pm(\xi)\right) \quad \text{for all} \quad \xi \in \mathbb{R}^n\backslash 0, \tag{1.15}$$

$$\inf_{|\xi|=1, j\neq k} \left|\varphi_j^\pm(\xi) - \varphi_k^\pm(\xi)\right| \geq d > 0. \tag{1.16}$$

Then the following theorem assures the existence of wave operators.

Theorem 1.2 *Let $a_{jk}^\pm(\xi)$ be entries of $A_\pm(\xi)$ satisfying* (1.15)–(1.16)*. Suppose that the $a_{jk}(t,\xi)$, $j,k = 1,\dots,m$, are positively homogeneous of order one in ξ, and satisfy* (1.3)–(1.5) *in such a way that*

$$a_{jk}(t,\xi/|\xi|) \longrightarrow a_{jk}^\pm(\xi/|\xi|) \text{ uniformly in } \xi \in \mathbb{R}^n\backslash 0 \text{ as } t \longrightarrow \pm\infty.$$

Assume also that the improper Riemann integrals of $\varphi_j(t,\xi) - \varphi_j^\pm(\xi)$ exist for each $\xi \in \mathbb{R}^n\backslash 0$. Then for each solution $V_-(t,x) \in C\big(\mathbb{R}; \left(L^2(\mathbb{R}^n)\right)^m\big)$ to (1.13)–(1.14)*, there exists a unique solution $U(t,x) \in C\big(\mathbb{R}; \left(L^2(\mathbb{R}^n)\right)^m\big)$ of*

$$D_t U = A(t,D_x)U$$

such that

$$\|V_-(t) - U(t)\|_{(L^2(\mathbb{R}^n))^m} \longrightarrow 0 \quad (t \longrightarrow -\infty).$$

For a corresponding $V_+(t,x)$ the same conclusion with $\pm$ reversed holds. Moreover, the mappings (*wave operators*)

$$\mathscr{W}_\pm : V_\pm(0,\cdot) \longmapsto U(0,\cdot)$$

are well-defined and bounded on $\left(L^2(\mathbb{R}^n)\right)^m$.

As a consequence of Theorems 1.1–1.2, we can construct scattering operators. More precisely, we have:

Corollary 1.3 *Let $\mathscr{W}_+^{-1}$ and $\mathscr{W}_-$ be as in Theorems* 1.1 *and* 1.2*, respectively. Assume that $a_{jk}^+(\xi) = a_{jk}^-(\xi)$ for $j, k = 1, \dots, m$. Then the mapping*

$$S = \mathscr{W}_+^{-1} \mathscr{W}_- : V_-(0,\cdot) \longmapsto V_+(0,\cdot)$$

defines a scattering operator, and it is bijective and bounded on $\left(L^2(\mathbb{R}^n)\right)^m$.

In Corollary 1.3, the operator $S : V_-(0,\cdot) \mapsto V_+(0,\cdot)$ is bijective and bounded on $\left(L^2(\mathbb{R}^n)\right)^m$ also without the assumption $a_{jk}^+(\xi) = a_{jk}^-(\xi)$ for $j, k = 1, \dots, m$. The reason to impose this condition is to be able to call S a scattering operator, with a physical meaning, that the initial state $V_-(t, x)$ and the final state $V_+(t, x)$ obey the same PDE whereas initial conditions do not matter.

Finally, let us look at some examples of settings to which our theorems apply. We note that although the equations may be of high order, it is important that we impose conditions only on one time-derivative of the coefficients. This is of crucial importance to being able to apply the obtained results to the Kirchhoff equations.

Our first example deals with higher order scalar equations.

Example 1.4 Consider the Cauchy problem for the m^{th} order strictly hyperbolic equation

$$L(t, D_t, D_x)u \equiv D_t^m u + \sum_{\substack{|\nu|+j=m \\ j \le m-1}} a_{\nu,j}(t) D_x^\nu D_t^j u = 0, \quad t \neq 0,$$

with Cauchy data

$$D_t^k u(x, 0) = f_k(x) \in H^{m-1-k}(\mathbb{R}^n), \quad k = 0, 1, \dots, m-1, \quad x \in \mathbb{R}^n,$$

where $D_t = -i\partial_t$ and $D_x^\nu = (-i\partial_{x_1})^{\nu_1} \cdots (-i\partial_{x_n})^{\nu_n}$, $i = \sqrt{-1}$, for $\nu = (\nu_1, \dots, \nu_n)$. We assume that $a_{\nu,j}(t)$ belong to $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$ and satisfy

$$a_{\nu,j}'(t) \in L^1(\mathbb{R}) \quad \text{for all} \quad \nu, j,$$

and the symbol $L(t, \tau, \xi)$ of the operator $L(t, D_t, D_x)$ has real roots $\varphi_1(t, \xi), \dots, \varphi_m(t, \xi)$ which are uniformly distinct for $\xi \neq 0$, i.e.,

$$L(t, \tau, \xi) = (\tau - \varphi_1(t, \xi)) \cdots (\tau - \varphi_m(t, \xi)),$$

$$\inf_{\substack{|\xi|=1, t \in \mathbb{R} \\ j \neq k}} |\varphi_j(t, \xi) - \varphi_k(t, \xi)| \ge d > 0.$$

The reference equation is

$$D_t^m v_\pm + \sum_{\substack{|\nu|+j=m \\ j \le m-1}} a_{\nu,j}^\pm D_x^\nu D_t^j v_\pm = 0,$$

with $a_{\nu,j}^\pm = \lim_{t \to \pm\infty} a_{\nu,j}(t)$, and the energy space is $\dot{H}^{m-1}(\mathbb{R}^n) \times \cdots \times L^2(\mathbb{R}^n)$.

The following example deals with coupled second order equations.

Example 1.5 Let us consider the coupled system of Cauchy problems

$$\begin{cases} \partial_t^2 u - c_1(t)^2 \Delta u + P_1(t, D_x) v = 0, \\ \partial_t^2 v - c_2(t)^2 \Delta v + P_2(t, D_x) u = 0, \end{cases}$$

for some second order homogeneous differential operators $P_1(t, D_x), P_2(t, D_x)$ which may depend on time, where we assume that

$$\left.\begin{aligned} c_k(t) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}), \quad & |\xi|^{-2} P_k(t, \xi) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}; L^\infty(\mathbb{R}^n \backslash 0)) \\ c_k'(t) \in L^1(\mathbb{R}), \quad & |\xi|^{-2} \partial_t P_k(t, \xi) \in L^1(\mathbb{R}; L^\infty(\mathbb{R}^n \backslash 0)) \end{aligned}\right\} \quad (k = 1, 2),$$

$$\inf_{t\in\mathbb{R},|\xi|=1}\left\{\left(c_1(t)^2-c_2(t)^2\right)^2+4P_1(t,\xi)P_2(t,\xi)\right\}>0,$$
$$\inf_{t\in\mathbb{R},|\xi|=1}\left\{c_1(t)^2c_2(t)^2-P_1(t,\xi)P_2(t,\xi)\right\}>0.$$

By taking the Fourier transform in the space variables and introducing the vector

$$V(t,\xi)={}^T(v_1(t,\xi),v_2(t,\xi),v_3(t,\xi),v_4(t,\xi))={}^T\left(|\xi|\widehat{u}(t,\xi),\widehat{u}'(t,\xi),|\xi|\widehat{v}(t,\xi),\widehat{v}'(t,\xi)\right)$$

we obtain the system

$$\frac{\partial V}{\partial t}=i\begin{pmatrix}0&-i|\xi|&0&0\\ ic_1(t)^2|\xi|&0&iP_1(t,\xi)|\xi|^{-1}&0\\ 0&0&0&-i|\xi|\\ iP_2(t,\xi)|\xi|^{-1}&0&ic_2(t)^2|\xi|&0\end{pmatrix}V=:iA(t,\xi)V.$$

The four characteristic roots of $\det(\tau I-A(t,\xi))=0$ in τ are given by

$$\varphi_{1,2,3,4}(t,\xi)=\pm\frac{|\xi|}{\sqrt{2}}\sqrt{c_1(t)^2+c_2(t)^2\pm\sqrt{\{c_1(t)^2-c_2(t)^2\}^2+4P_1(t,\xi)P_2(t,\xi)|\xi|^{-4}}}.$$

The reference system is

$$\begin{cases}\partial_t^2u_\pm-c_{1,\pm}^2\Delta u_\pm+P_{1,\pm}(D_x)v_\pm=0,\\ \partial_t^2v_\pm-c_{2,\pm}^2\Delta v_\pm+P_{2,\pm}(D_x)u_\pm=0,\end{cases}$$

with the limits

$$c_{k,\pm}=\lim_{t\to\pm\infty}c_k(t),\quad |\xi|^{-2}P_{k,\pm}(\xi)=\lim_{t\to\pm\infty}|\xi|^{-2}P_k(t,\xi)$$

uniformly in $\xi\in\mathbb{R}^n\backslash 0$ for $k=1,2$, and the energy space is $\left(\dot{H}^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\right)^2$.

We conclude this section by stating our plan. In Section 2 we will find a representation formula for (1.1)–(1.2). The proof of Theorem 1.1 will be given in Section 3 and Section 4 separately. In the last section we will prove Theorem 1.2.

2 Representation formulae via asymptotic integrations

In this section we will derive a representation formula for (1.1) along the method developed in [8]. Let us first analyse certain basic properties of characteristic roots $\varphi_k(t,\xi)$ of (1.4).

Proposition 2.1 *Let the operator $D_t-A(t,D_x)$ satisfy assumptions* (1.4)–(1.5). *If the $a_{jk}(t,\xi/|\xi|)$ belong to* $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R};L^\infty(\mathbb{R}^n\backslash 0))$ *for $j,k=1,\dots,m$, then each $\partial_t\varphi_k(t,\xi)$, $k=1,\dots,m$, is positively homogeneous of order one in ξ. In addition, if $\partial_t a_{jk}(t,\xi/|\xi|)$ belong to $L^1(\mathbb{R};L^\infty(\mathbb{R}^n\backslash 0))$ for $j,k=1,\dots,m$, then we have also $\partial_t\varphi_k(\cdot,\xi/|\xi|)\in L^1(\mathbb{R};L^\infty(\mathbb{R}^n\backslash 0))$. Furthermore, there exist functions $\varphi_k^\pm(\xi)$, $k=1,\dots,m$, positively homogeneous of order one, such that*

$$\varphi_k(t,\xi)\longrightarrow\varphi_k^\pm(\xi)\ \textit{uniformly on}\ |\xi|=1\ \textit{as}\ t\longrightarrow\pm\infty. \tag{2.1}$$

Proof. Let us observe first that $\varphi_k(t,\xi)$ are bounded with respect to $t\in\mathbb{R}$, i.e.,

$$|\varphi_k(t,\xi)|\le C|\xi|,\quad \text{for all}\quad \xi\in\mathbb{R}^n,\quad t\in\mathbb{R},\quad k=1,\dots,m. \tag{2.2}$$

For this, we use the fact that $\varphi_k(t,\xi)$ are the roots of polynomial $L(t,\tau,\xi)=\det(\tau I-A(t,\xi))$ of the form

$$L(t,\tau,\xi)=\tau^m+\alpha_1(t,\xi)\tau^{m-1}+\cdots+\alpha_m(t,\xi)$$

with $|\alpha_j(t,\xi)| \le M|\xi|^j$, for some $M \ge 1$, and the proof is elementary (see Proposition 2.3 in [8]). Here we put

$$\alpha_k(t,\xi) = (-1)^k \sum_{i_1<i_2<\cdots<i_k} \det \begin{pmatrix} a_{i_1 i_1}(t,\xi) & \cdots & a_{i_1 i_k}(t,\xi) \\ \vdots & \ddots & \vdots \\ a_{i_k i_1}(t,\xi) & \cdots & a_{i_k i_k}(t,\xi) \end{pmatrix}.$$

Differentiating (1.4) with respect to t, we get

$$\frac{\partial L(t,\tau,\xi)}{\partial t} = \sum_{j=0}^{m-1} \partial_t \alpha_{m-j}(t,\xi)\tau^j = -\sum_{k=1}^{m} \partial_t \varphi_k(t,\xi) \prod_{r\neq k} (\tau - \varphi_r(t,\xi)).$$

Setting $\tau = \varphi_k(t,\xi)$, we obtain

$$\partial_t \varphi_k(t,\xi) \prod_{r\neq k} (\varphi_k(t,\xi) - \varphi_r(t,\xi)) = -\sum_{j=0}^{m-1} \partial_t \alpha_{m-j}(t,\xi)\varphi_k(t,\xi)^j. \tag{2.3}$$

The positive homogeneity of order one of $\partial_t \varphi_k(t,\xi)$ is an immediate consequence of (2.3). Now, by using (1.5), (2.2), and the assumption that $|\xi|^{-j}\partial_t \alpha_j(\cdot,\xi) \in L^1(\mathbb{R}; L^\infty(\mathbb{R}^n\backslash 0))$ for all j, we conclude that $\partial_t \varphi_k(\cdot, \xi/|\xi|) \in L^1(\mathbb{R}; L^\infty(\mathbb{R}^n\backslash 0))$ for $k = 1,\dots,m$.

Finally, setting

$$\varphi_k^{\pm}(\xi) = \varphi_k(0,\xi) + \int_0^{\pm\infty} \partial_t \varphi_k(t,\xi)\, dt,$$

we get (2.1). The proof is complete. □

We prepare the next lemma.

Lemma 2.2 ([11] Proposition 6.4.) *Assume* (1.3)–(1.5). *Then there exists a matrix* $\mathscr{N} = \mathscr{N}(t,\xi)$ *positively homogeneous of order* 0 *in* ξ *satisfying the following properties*:

(i) $\mathscr{N}(t,\xi)A(t,\xi/|\xi|) = \mathscr{D}(t,\xi)\mathscr{N}(t,\xi)$, *where*

$$\mathscr{D}(t,\xi) = \mathrm{diag}\,(\varphi_1(t,\xi/|\xi|),\dots,\varphi_m(t,\xi/|\xi|));$$

(ii) $\inf_{\xi\in\mathbb{R}^n\backslash 0, t\in\mathbb{R}} |\det \mathscr{N}(t,\xi))| > 0$;

(iii) $\mathscr{N}(t,\xi) \in \mathrm{Lip}_{\mathrm{loc}}\big(\mathbb{R}; (L^\infty(\mathbb{R}^n \setminus 0))^{m^2}\big)$ *and* $\partial_t \mathscr{N}(t,\xi) \in L^1\big(\mathbb{R}; (L^\infty(\mathbb{R}^n\backslash 0))^{m^2}\big)$.

(iv) *There exists the uniform limits* $\mathscr{N}^{\pm}(\xi)$ *of* $\mathscr{N}(t,\xi)$ *as* $t \to \pm\infty$:

$$\mathscr{N}^{\pm}(\xi) = \lim_{t\to\pm\infty} \mathscr{N}(t,\xi) \textit{ uniformly in } \xi \in \mathbb{R}^n\backslash 0.$$

In fact, Mizohata considered the case where the symbol of a differential system depends on the space-time variable (x,t) in Proposition 6.4 of [11]. Hence the corresponding diagonaliser $\mathcal{N}(t,x,\xi)$ is defined there on a bounded interval $[0,T]$, whereas the present case does not depend on x, so the diagonaliser $\mathcal{N}(t,\xi)$ is globally defined on the whole of $\mathbb{R}$.

Let us find a representation formula for solutions to (1.1). Applying the Fourier transform to (1.1), we get the following ordinary differential system:

$$D_t \boldsymbol{v} = A(t,\xi/|\xi|)|\xi|\boldsymbol{v}, \quad \boldsymbol{v} = \widehat{U}, \tag{2.4}$$

where $\widehat{U} = \widehat{U}(t,\xi)$ stands for the Fourier transform of $U(t,x)$ in x on $\mathbb{R}^n$. We proceed with the argument by following Ascoli [2] and Wintner [16], (cf. Coddington & Levinson [3] and Hartman [4]). Multiplying (2.4) by $\mathscr{N} = \mathscr{N}(t,\xi)$ from Lemma 2.2 and putting $\mathscr{N}\boldsymbol{v} = \boldsymbol{w}$, we get

$$D_t \boldsymbol{w} = \mathscr{D}|\xi|\boldsymbol{w} + (D_t\mathscr{N})\boldsymbol{v} = \left(\mathscr{D}|\xi| + (D_t\mathscr{N})\mathscr{N}^{-1}\right)\boldsymbol{w}, \tag{2.5}$$

since

$$\mathscr{N} A(t, \xi/|\xi|) = \mathscr{D}\mathscr{N}$$

by Lemma 2.2. Let us write solutions of (2.5) by using a solution of

$$D_t \boldsymbol{y} = \mathscr{D}|\xi|\boldsymbol{y}. \tag{2.6}$$

To this end, let $\Phi(t,\xi)$ be the fundamental matrix of (2.6), i.e.,

$$\Phi(t,\xi) = \operatorname{diag}\left(e^{i\int_0^t \varphi_1(s,\xi)\,ds}, \cdots, e^{i\int_0^t \varphi_m(s,\xi)\,ds}\right). \tag{2.7}$$

If we perform the Wronskian transform

$$\boldsymbol{a}(t,\xi) = \Phi(t,\xi)^{-1}\boldsymbol{w}(t,\xi),$$

then system (2.5) reduces to a system

$$D_t \boldsymbol{a} = C(t,\xi)\boldsymbol{a},$$

where $C(t,\xi)$ is given by

$$C(t,\xi) = \Phi(t,\xi)^{-1}(D_t\mathscr{N}(t,\xi))\mathscr{N}(t,\xi)^{-1}\Phi(t,\xi). \tag{2.8}$$

We note that $C(t,\xi) \in L^1\left(\mathbb{R}; (L^\infty(\mathbb{R}^n\backslash 0))^{m^2}\right)$, since $D_t\mathscr{N}(t,\xi) \in L^1\left(\mathbb{R}; (L^\infty(\mathbb{R}^n\backslash 0))^{m^2}\right)$ by Lemma 2.2. Since $\boldsymbol{w}(t,\xi) = \Phi(t,\xi)\boldsymbol{a}(t,\xi)$ and $\mathscr{N}(t,\xi)\boldsymbol{v}(t,\xi) = \boldsymbol{w}(t,\xi)$, we get

$$\boldsymbol{v}(t,\xi) = \mathscr{N}(t,\xi)^{-1}\Phi(t,\xi)\boldsymbol{a}(t,\xi).$$

Now let $(\boldsymbol{v}_0(t,\xi), \ldots, \boldsymbol{v}_{m-1}(t,\xi))$ be the fundamental matrix of (2.4). This means, in particular, that

$$(\boldsymbol{v}_0(0,\xi), \ldots, \boldsymbol{v}_{m-1}(0,\xi)) = I.$$

Then each $\boldsymbol{v}_j(t,\xi)$ can be represented by

$$\boldsymbol{v}_j(t,\xi) = \mathscr{N}(t,\xi)^{-1}\Phi(t,\xi)\boldsymbol{a}^j(t,\xi),$$

where $\boldsymbol{a}^j(t,\xi)$ are the corresponding amplitude functions to $\boldsymbol{v}_j(t,\xi)$. Since

$$\widehat{U}(t,\xi) = \sum_{j=0}^{m-1} \boldsymbol{v}_j(t,\xi)\widehat{f}_j(\xi),$$

we arrive at

$$\widehat{U}(t,\xi) = \sum_{j=0}^{m-1} \mathscr{N}(t,\xi)^{-1}\Phi(t,\xi)\boldsymbol{a}^j(t,\xi)\widehat{f}_j(\xi).$$

Finally, let us find estimates for $\boldsymbol{a}^j(t,\xi)$. Recalling that $\boldsymbol{a}^j(t,\xi)$ satisfy the system

$$D_t\boldsymbol{a}^j = C(t,\xi)\boldsymbol{a}^j \quad \text{with} \quad \left(\boldsymbol{a}^0(0,\xi), \ldots, \boldsymbol{a}^{m-1}(0,\xi)\right) = \mathscr{N}(0,\xi),$$

we can write $\boldsymbol{a}^j(t,\xi)$ by the Picard series:

$$\boldsymbol{a}^j(t,\xi) = \left(I + i\int_0^t C(\tau_1,\xi)\,d\tau_1 + i^2\int_0^t C(\tau_1,\xi)\,d\tau_1\int_0^{\tau_1} C(\tau_2,\xi)\,d\tau_2 + \cdots\right)\boldsymbol{a}^j(0,\xi).$$

This implies that

$$\left|\boldsymbol{a}^j(t,\xi)\right| \le e^{c\int_{\mathbb{R}} \|\partial_t \mathscr{N}(s,\xi)\|_{(L^\infty(\mathbb{R}^n\backslash 0))^{m^2}}\,ds} \left|\boldsymbol{a}^j(0,\xi)\right|, \tag{2.9}$$

where we have used the following fact: Let $f(t)$ be a locally integrable function on $\mathbb{R}$. Then

$$e^{\int_s^t f(\tau)\,d\tau} = 1 + \int_s^t f(\tau_1)\,d\tau_1 + \int_s^t f(\tau_1)\,d\tau_1 \int_s^{\tau_1} f(\tau_2)\,d\tau_2 + \cdots.$$

By using equations $D_t\boldsymbol{a}^j = C(t,\xi)\boldsymbol{a}^j$, the estimates (2.9), and the fact that $C(t,\xi) \in L^1\left(\mathbb{R};(L^\infty(\mathbb{R}^n\backslash 0))^{m^2}\right)$, we conclude that $D_t\boldsymbol{a}^j(\cdot,\xi) \in L^1(\mathbb{R};(L^\infty(\mathbb{R}^n\backslash 0))^m)$; thus there exist the limits

$$\lim_{t\to\pm\infty} \boldsymbol{a}^j(t,\xi) = \boldsymbol{\alpha}^j_\pm(\xi) \text{ uniformly in } \xi \in \mathbb{R}^n\backslash 0.$$

With the above argument in mind, we can obtain the main result in this section.

Proposition 2.3 *Assume* (1.3)–(1.5). *Let $\mathscr{N}(t,\xi)$ be the diagonaliser of $A(t,\xi/|\xi|)$ constructed in Lemma 2.2, and $\Phi(t,\xi)$ as in* (2.7):

$$\Phi(t,\xi) = \operatorname{diag}\left(e^{i\int_0^t \varphi_1(s,\xi)\,ds},\cdots,e^{i\int_0^t \varphi_m(s,\xi)\,ds}\right).$$

Then there exist vector-valued functions $\boldsymbol{a}^j(t,\xi)$, $j = 0,1,\ldots,m-1$, determined by initial value problems

$$D_t\boldsymbol{a}^j(t,\xi) = C(t,\xi)\boldsymbol{a}^j(t,\xi), \quad \left(\boldsymbol{a}^0(0,\xi),\ldots,\boldsymbol{a}^{m-1}(0,\xi)\right) = \mathscr{N}(0,\xi),$$

$$\textit{with} \quad C(t,\xi) = \Phi(t,\xi)^{-1}(D_t\mathscr{N}(t,\xi))\mathscr{N}(t,\xi)^{-1}\Phi(t,\xi) \in L^1\left(\mathbb{R};(L^\infty(\mathbb{R}^n\backslash 0))^{m^2}\right),$$

such that the solution $U(t,x)$ of (1.1) *is represented by*

$$U(t,x) = \sum_{j=0}^{m-1} \mathscr{F}^{-1}\left[\mathscr{N}(t,\xi)^{-1}\Phi(t,\xi)\boldsymbol{a}^j(t,\xi)\widehat{f}_j(\xi)\right](x). \tag{2.10}$$

Here $\mathscr{F}^{-1}[\cdot]$ stands for the inverse Fourier transform on $\mathbb{R}^n$. Moreover, there exists a constant $c > 0$ such that

$$\sup_{t\in\mathbb{R}} \left\|\boldsymbol{a}^j(t,\xi)\right\|_{(L^\infty(\mathbb{R}^n\backslash 0))^m} \le c, \quad j = 0,1,\ldots,m-1,$$

and the limits

$$\lim_{t\to\pm\infty} \boldsymbol{a}^j(t,\xi) = \boldsymbol{\alpha}^j_\pm(\xi), \quad j = 0,1,\ldots,m-1,$$

exist uniformly in $\xi \in \mathbb{R}^n\backslash 0$.

Proposition 2.3 is also known as Levinson's lemma (see Coddington and Levinson [3]) in the theory of ordinary differential equations; the new feature here is the additional dependence on ξ tracing which is crucial for our analysis.

3 Proof of Theorem 1.1 (i)

We recall the assumption that

$$\Theta_j^\pm(\xi) = \int_0^{\pm\infty} \Phi_{j,\pm}(s,\xi)\,ds \text{ exists for each } \xi \in \mathbb{R}^n\backslash 0, \tag{3.1}$$

where

$$\Phi_{j,\pm}(s,\xi) = \varphi_j(s,\xi) - \varphi_j^\pm(\xi).$$

Writing

$$\int_0^t \varphi_j(s,\xi)\,ds = \varphi_j^{\pm}(\xi)t + \Theta_j^{\pm}(\xi) - \int_t^{\pm\infty} \Phi_{j,\pm}(s,\xi)\,ds,$$

we have

$$e^{i\int_0^t \varphi_j(s,\xi)\,ds} = e^{i\left(\varphi_j^{\pm}(\xi)t+\Theta_j^{\pm}(\xi)\right)} + \Psi_{j,\pm}(t,\xi), \tag{3.2}$$

where

$$\Psi_{j,\pm}(t,\xi) = e^{i\left(\varphi_j^{\pm}(\xi)t+\Theta_j^{\pm}(\xi)\right)}\left(\exp\left(-i\int_t^{\pm\infty}\Phi_{j,\pm}(s,\xi)\,ds\right)-1\right).$$

Here we note from (3.1) that

$$\Psi_{j,\pm}(t,\xi) \longrightarrow 0 \text{ for each } \xi\in\mathbb{R}^n\backslash 0 \text{ as } t\longrightarrow \pm\infty.$$

Putting

$$\begin{cases} \Phi_{\pm}(t,\xi) = \operatorname{diag}\left(e^{i\varphi_1^{\pm}(\xi)t},\dots,e^{i\varphi_m^{\pm}(\xi)t}\right),\\ D_{\pm}(\xi) = \operatorname{diag}\left(e^{i\Theta_1^{\pm}(\xi)},\dots,e^{i\Theta_m^{\pm}(\xi)}\right),\\ \Psi_{\pm}(t,\xi) = \operatorname{diag}\left(\Psi_{1,\pm}(t,\xi),\dots,\Psi_{m,\pm}(t,\xi)\right), \end{cases}$$

we can write (3.2) as

$$\begin{aligned} &\Phi(t,\xi) = \Phi_{\pm}(t,\xi)D_{\pm}(\xi) + \Psi_{\pm}(t,\xi) \quad \text{with}\\ &\Psi_{\pm}(t,\xi) \longrightarrow O \text{ for each } \xi\in\mathbb{R}^n\backslash 0 \text{ as } t\longrightarrow\pm\infty. \end{aligned} \tag{3.3}$$

Hence, plugging (3.3) into (2.10) from Proposition 2.3, we find that

$$\begin{aligned} U(t,x) &= \sum_{j=0}^{m-1}\mathscr{F}^{-1}\left[\mathscr{N}_{\pm}(\xi)^{-1}\Phi_{\pm}(t,\xi)D_{\pm}(\xi)\boldsymbol{\alpha}_{\pm}^j(\xi)\widehat{f}_j(\xi)\right](x)\\ &+\sum_{j=0}^{m-1}\mathscr{F}^{-1}\left[\left(\mathscr{N}(t,\xi)^{-1}-\mathscr{N}_{\pm}(\xi)^{-1}\right)\Phi_{\pm}(t,\xi)D_{\pm}(\xi)\boldsymbol{\alpha}_{\pm}^j(\xi)\widehat{f}_j(\xi)\right](x)\\ &+\sum_{j=0}^{m-1}\mathscr{F}^{-1}\left[\mathscr{N}_{\pm}(\xi)^{-1}\Phi_{\pm}(t,\xi)D_{\pm}(\xi)\left(\boldsymbol{a}^j(t,\xi)-\boldsymbol{\alpha}_{\pm}^j(\xi)\right)\widehat{f}_j(\xi)\right](x)\\ &+\sum_{j=0}^{m-1}\mathscr{F}^{-1}\left[\mathscr{N}(t,\xi)^{-1}\Psi_{\pm}(t,\xi)\boldsymbol{a}^j(t,\xi)\widehat{f}_j(\xi)\right](x), \quad t\gtrless 0. \end{aligned}$$

It can be verified that

$$V_{\pm}(t,x) := \sum_{j=0}^{m-1}\mathscr{F}^{-1}\left[\mathscr{N}_{\pm}(\xi)^{-1}\Phi_{\pm}(t,\xi)D_{\pm}(\xi)\boldsymbol{\alpha}_{\pm}^j(\xi)\widehat{f}_j(\xi)\right](x) \tag{3.4}$$

satisfy equations (1.7). Thus, by using (3.3) and the following convergences:

$$\mathscr{N}(t,\xi)^{-1}\longrightarrow\mathscr{N}_{\pm}(\xi)^{-1}, \quad \boldsymbol{a}^j(t,\xi)\longrightarrow\boldsymbol{\alpha}_{\pm}^j(\xi) \text{ uniformly in } \xi\in\mathbb{R}^n\backslash 0 \text{ as } t\longrightarrow\pm\infty,$$

we conclude from Plancherel's identity and Lebesgue's dominated convergence theorem that

$$\|U(t,\cdot)-V_{\pm}(t,\cdot)\|_{(L^2(\mathbb{R}^n))^m}\longrightarrow 0 \quad (t\longrightarrow\pm\infty).$$

In conclusion, $U(x,t)$ is asymptotically free. Moreover, the mappings

$$\mathscr{W}_{\pm}^{-1}: U(0,\cdot)\longmapsto V_{\pm}(0,\cdot) = \sum_{j=0}^{m-1}\mathscr{F}^{-1}\left[\mathscr{N}_{\pm}(\xi)^{-1}D_{\pm}(\xi)\boldsymbol{\alpha}_{\pm}^j(\xi)\widehat{f}_j(\xi)\right](\cdot)$$

are bijective and bounded on $\left(L^2(\mathbb{R}^n)\right)^m$. The part (i) of Theorem 1.1 is thus proved. □

4 Proof of Theorem 1.1 (ii)

Observing from representation formulae (2.10) and (3.4) of $U(t,x)$ and $V_\pm(t,x)$, respectively, we can write, by using Plancherel's identity,

$$\widehat{U}(t,\xi) = \sum_{j=0}^{m-1}\sum_{k=1}^{m}\int_{\mathbb{R}^n} \boldsymbol{a}_k^{(j)}(t,\xi)e^{i\int_0^t \varphi_k(s,\xi)\,ds}\widehat{f}_j(\xi)\,d\xi,$$

$$\widehat{V}_\pm(t,\xi) = \sum_{j=0}^{m-1}\sum_{k=1}^{m}\int_{\mathbb{R}^n} \boldsymbol{a}_{k,\pm}^{(j)}(\xi)e^{i\varphi_k^\pm(\xi)t}\widehat{f}_j(\xi)\,d\xi,$$

where $\boldsymbol{a}_k^{(j)}(t,\xi)$ and $\boldsymbol{a}_{k,\pm}^{(j)}(\xi)$ are m-row vector-valued functions satisfying

$$\sup_{t\in\mathbb{R},\xi\in\mathbb{R}^n\setminus 0}\left|\boldsymbol{a}_k^{(j)}(t,\xi)\right| \le C \quad\text{and}\quad \sup_{\xi\in\mathbb{R}^n\setminus 0}\left|\boldsymbol{a}_{k,\pm}^{(j)}(\xi)\right| \le C.$$

Hence

$$\boldsymbol{a}_k^{(j)}(t,\xi)\widehat{f}_j(\xi),\, \boldsymbol{a}_{k,\pm}^{(j)}(\xi)\widehat{f}_j(\xi) \in \big(L^2(\mathbb{R}^n)\big)^m.$$

The next lemma is part of the subsequent argument.

Lemma 4.1 *Let $\vartheta(t,\xi)$ be a real-valued function on $\mathbb{R}\times\mathbb{R}^n$ which is positively homogeneous of order one in ξ, and satisfies $\vartheta(t,\xi/|\xi|)\to\infty$ for a.e. $\xi\in\mathbb{R}^n\setminus 0$ as $t\to\infty$. Assume that a function $a(t,\xi)$ belongs to $L^\infty\big(\mathbb{R};L^1(\mathbb{R}^n)\big)$ and satisfies*

$$\sup_{t\in\mathbb{R}}|a(t,\xi)| \le m(|\xi|) \tag{4.1}$$

for some radially symmetric function $m(\xi)=m(|\xi|)\in L^1(\mathbb{R}^n)$. Then

$$I(t) := \int_{\mathbb{R}^n} e^{i\vartheta(t,\xi)}a(t,\xi)\,d\xi \longrightarrow 0 \quad \text{as} \quad t\longrightarrow\infty.$$

Proof. Making change of variable $\xi=\rho\omega$ ($\rho=|\xi|$, $\omega=\xi/|\xi|\in\mathbb{S}^{n-1}$), we get

$$I(t) = \int_{\mathbb{S}^{n-1}}\left(\int_0^\infty e^{i\rho\vartheta(t,\omega)}a(t,\rho\omega)\rho^{n-1}\,d\rho\right)d\sigma(\omega),$$

where $d\sigma(\omega)$ is the $(n-1)$-dimensional Hausdorff measure. By using the assumption that $a(t,\xi)$ belongs to $L^\infty(\mathbb{R};L^1(\mathbb{R}^n))$, we see from Fubini's theorem that $a(t,\rho\omega)\rho^{n-1}\in L^\infty\big(\mathbb{R};L^1(0,\infty)\big)$. Since $\vartheta(t,\omega)\to\infty$ for a.e. $\omega\in\mathbb{S}^{n-1}$ as $t\to\infty$, extending the function a by zero to $\rho<0$ and applying Riemann-Lebesgue's lemma, we conclude that for a.e. $\omega\in\mathbb{S}^{n-1}$,

$$\int_0^\infty e^{i\rho\vartheta(t,\omega)}a(t,\rho\omega)\rho^{n-1}\,d\rho \longrightarrow 0 \quad\text{as}\quad t\longrightarrow\infty.$$

Moreover, by using (4.1), we have

$$\left|\int_0^\infty e^{i\rho\vartheta(t,\omega)}a(t,\rho\omega)\rho^{n-1}\,d\rho\right| \le \int_0^\infty m(\rho)\rho^{n-1}\,d\rho \in L^1\big(\mathbb{S}^{n-1}\big).$$

Thus, this estimate together with the previous convergence imply, by using Lebesgue's dominated convergence theorem, that $I(t)\to 0$ as $t\to\infty$. This proves the lemma. □

The proof of part (ii) in Theorem 1.1 is reduced to the next lemma provided that $(f_0(x),\dots,f_{m-1}(x))$ are non-trivial and radially symmetric.

Lemma 4.2 *Let $\varphi_j(t,\xi)$ and $\varphi_j^{\pm}(\xi)$, $j = 1, \dots, m$, be the phase functions as in* (1.5) *and* (1.9)*, respectively. Suppose that*

$$|\vartheta_j(t,\xi) - \varphi_j^{\pm}(\xi)t| \longrightarrow +\infty \text{ for a.e. } \xi \in \mathbb{R}^n \backslash 0 \text{ as } t \longrightarrow \pm\infty, \tag{4.2}$$

where $\vartheta_j(t,\xi) = \int_0^t \varphi_j(s,\xi)\,ds$. Let $A_j(t,\xi), B_j(t,\xi) \in C\big(\mathbb{R}; L^2(\mathbb{R}^n)\big)$, $j = 1, \dots, m-1$, satisfy

$$A_j(t,\xi) \longrightarrow A_j^{\pm}(\xi), \quad B_j(t,\xi) \longrightarrow B_j^{\pm}(\xi) \text{ uniformly in } \xi \text{ as } t \longrightarrow \pm\infty,$$

with some $A_j^{\pm}(\xi), B_j^{\pm}(\xi) \in L^2(\mathbb{R}^n)$. Assume also that $A_j(t,\xi)$ and $B_j(t,\xi)$ are bounded by some radially symmetric functions $M_j(|\xi|)$ and $N_j(|\xi|)$ from $L^2(\mathbb{R}^n)$;

$$|A_j(t,\xi)| \le M_j(|\xi|) \quad \text{and} \quad |B_j(t,\xi)| \le N_j(|\xi|), \quad j = 1, \dots, m-1.$$

Then we have

$$\left\| \sum_{j=1}^m \left\{ A_j(t,\xi) e^{i\vartheta_j(t,\xi)} - B_j(t,\xi) e^{i\varphi_j^{\pm}(\xi)t} \right\} \right\|_{L^2(\mathbb{R}^n)} \longrightarrow \left\{ \sum_{j=1}^m \left(\left\|A_j^{\pm}(\xi)\right\|_{L^2(\mathbb{R}^n)}^2 + \left\|B_j^{\pm}(\xi)\right\|_{L^2(\mathbb{R}^n)}^2 \right) \right\}^{1/2}$$

as $t \to \pm\infty$.

Proof. Putting

$$K_{\pm}(t,\xi) = \sum_{j=1}^m \left\{ A_j(t,\xi) e^{i\vartheta_j(t,\xi)} - B_j(t,\xi) e^{i\varphi_j^{\pm}(\xi)t} \right\},$$

we can write

$$|K_{\pm}(t,\xi)|^2 = \sum_{j=1}^m \left(|A_j(t,\xi)|^2 + |B_j(t,\xi)|^2 \right) + \operatorname{Re} H_{\pm}(t,\xi), \tag{4.3}$$

where

$$\begin{aligned} H_{\pm}(t,\xi) = {} & 2\sum_{j<k} \left\{ e^{i\{\vartheta_j(t,\xi) - \vartheta_k(t,\xi)\}} A_j(t,\xi)\overline{A_k(t,\xi)} + e^{i\{\varphi_j^{\pm}(\xi)t - \varphi_k^{\pm}(\xi)t\}} B_j(t,\xi)\overline{B_k(t,\xi)} \right\} \\ & - 2\sum_{j=1}^m e^{i\{\vartheta_j(t,\xi) - \varphi_j^{\pm}(\xi)t\}} A_j(t,\xi)\overline{B_j(t,\xi)} - 2\sum_{j<k} e^{i\{\vartheta_j(t,\xi) - \varphi_k^{\pm}(\xi)t\}} A_j(t,\xi)\overline{B_k(t,\xi)}. \end{aligned}$$

We can check that all the phases in $H_{\pm}(t,\xi)$ are unbounded in t. Indeed, it follows from (1.5) and (1.9) that if $j < k$, then

$$|\vartheta_j(t,\xi) - \vartheta_k(t,\xi)| = \left| \int_0^t (\varphi_j(s,\xi) - \varphi_k(s,\xi))\,ds \right| \ge d|\xi||t| \longrightarrow +\infty, \tag{4.4}$$

$$\left|\varphi_j^{\pm}(\xi)t - \varphi_k^{\pm}(\xi)t\right| \ge d|\xi||t| \longrightarrow +\infty, \tag{4.5}$$

as $t \to \pm\infty$. Since $\varphi_j(t,\xi) \to \varphi_j^{\pm}(\xi)$ uniformly in $|\xi| = 1$ as $t \to \pm\infty$ by Proposition 2.1, it follows that for any $\varepsilon > 0$ there exists a number $T > 0$ such that

$$\left|\varphi_j(s,\xi) - \varphi_k^{\pm}(\xi)\right| \ge (d - \varepsilon)|\xi|, \quad j \ne k, \quad |s| > T, \quad \xi \ne 0.$$

Hence,

$$\begin{aligned}\left|\vartheta_j(t,\xi)-\varphi_k^{\pm}(\xi)t\right| &= \left|\int_0^t \left(\varphi_j(s,\xi)-\varphi_k^{\pm}(\xi)\right)\, ds\right| \\ &\geq \left|\int_T^t \left(\varphi_j(s,\xi)-\varphi_k^{\pm}(\xi)\right)\, ds\right| - \left|\int_0^T \left(\varphi_j(s,\xi)-\varphi_k^{\pm}(\xi)\right)\, ds\right| \\ &\geq (d-\varepsilon)|t-T||\xi| - \left|\int_0^T \left(\varphi_j(s,\xi)-\varphi_k^{\pm}(\xi)\right)\, ds\right| \longrightarrow +\infty \quad (t \longrightarrow \pm\infty).\end{aligned} \tag{4.6}$$

Thus by using Lemma 4.1, (4.2)–(4.6), the fact that $\varphi_j(s,\xi)-\varphi_k^{\pm}(\xi)$ does not change sign for s large enough by strict hyperbolicity, and the fact that the product of two functions in $L^2(\mathbb{R}^n)$ belongs to $L^1(\mathbb{R}^n)$, we get

$$\int_{\mathbb{R}^n} H_{\pm}(t,\xi)\, d\xi \longrightarrow 0$$

as $t \to \pm\infty$. In conclusion, we have

$$\int_{\mathbb{R}^n} |K_{\pm}(t,\xi)|^2\, d\xi \longrightarrow \sum_{j=0}^{m-1} \int_{\mathbb{R}^n} \left(\left|A_j^{\pm}(\xi)\right|^2 + \left|B_j^{\pm}(\xi)\right|^2\right) d\xi \quad (t \longrightarrow \pm\infty),$$

respectively. The proof of Lemma 4.2 is complete. □

5 Proof of Theorem 1.2

Let $V_{\pm} = V_{\pm}(t,x)$ be solutions to the Cauchy problems

$$D_t V_{\pm} = A_{\pm}(D_x)V_{\pm}, \quad x \in \mathbb{R}^n, \quad \pm t > 0,$$

with Cauchy data

$$V_{\pm}(0,x) = {}^T\left(f_0^{\pm}(x), \dots, f_{m-1}^{\pm}(x)\right).$$

Let $\mathscr{N}_{\pm}(\xi)$ be diagonalisers of $A_{\pm}(\xi/|\xi|)$, i.e.,

$$\mathscr{N}_{\pm}(\xi)A_{\pm}(\xi/|\xi|) = \mathscr{D}_{\pm}(\xi/|\xi|)\mathscr{N}_{\pm}(\xi),$$

where we put

$$\mathscr{D}_{\pm}(\xi) = \operatorname{diag}\left(\varphi_1^{\pm}(\xi), \dots, \varphi_m^{\pm}(\xi)\right).$$

Denoting by $\boldsymbol{e}^0, \dots, \boldsymbol{e}^{m-1}$ the standard unit vectors in $\mathbb{R}^m$, we can write

$$V_{\pm}(t,x) = \sum_{j=0}^{m-1} \mathscr{F}^{-1}\left[\mathscr{N}_{\pm}(\xi)^{-1}\Phi_{\pm}(t,\xi)\boldsymbol{e}^j \widehat{f}_j^{\pm}(\xi)\right](x).$$

We will find an asymptotic integration for

$$D_t\widehat{U}(t,\xi) = A(t,\xi)\widehat{U}(t,\xi),$$

such that $\widehat{U}(t,\xi)$ is asymptotic to $\widehat{V}_{\pm}(t,\xi)$ as $t \to \pm\infty$.

We shall prove the following:

Proposition 5.1 *Assume* (1.15)–(1.16). *Let $\mathscr{N}(t,\xi)$ be the diagonaliser of $A(t,\xi/|\xi|)$ from Lemma* 2.2, *and put*

$$\Phi(t,\xi) = \operatorname{diag}\left(e^{i\int_0^t \varphi_1(s,\xi)\,ds}, \dots, e^{i\int_0^t \varphi_m(s,\xi)\,ds}\right).$$

Then there exist fundamental matrices $W_\pm(t,\xi)$ of $D_t\widehat{U}(t,\xi) = A(t,\xi)\widehat{U}(t,\xi)$ such that

$$W_\pm(t,\xi) = \mathscr{N}(t,\xi)^{-1}\Phi(t,\xi)(I + R_\pm(t,\xi)) \quad \textit{with}$$

$$R_\pm(t,\xi) \longrightarrow O \textit{ uniformly in } \xi \in \mathbb{R}^n\backslash 0 \textit{ as } t \longrightarrow \pm\infty.$$

P r o o f. We prove the case "$-$", since the case "$+$" is similar. The idea of the proof resembles that of Hartman [4]. Similar to (2.8), we define a matrix $C_-(t,\xi)$ to be

$$C_-(t,\xi) = \Phi(t,\xi)^{-1}(D_t\mathscr{N}(t,\xi))\mathscr{N}(t,\xi)^{-1}\Phi(t,\xi). \tag{5.1}$$

For an arbitrarily fixed $\sigma \in \mathbb{R}$, let $\boldsymbol{a}_-^j(t,\xi;\sigma)$, $j = 0,\dots,m-1$, be solutions of the following problems:

$$D_t\boldsymbol{a}_-^j(t,\xi;\sigma) = C_-(t,\xi)\boldsymbol{a}_-^j(t,\xi;\sigma), \quad \boldsymbol{a}_-^j(\sigma,\xi;\sigma) = \boldsymbol{e}^j,$$

where $\boldsymbol{e}^j = {}^T(0,\dots,\overset{j}{\smile}{1},\dots,0)$. Hence $\boldsymbol{a}_-^j(t,\xi;\sigma)$ can be written as Picard series:

$$\boldsymbol{a}_-^j(t,\xi;\sigma) = \left(I + i\int_\sigma^t C_-(\tau_1,\xi)\,d\tau_1 + i^2\int_\sigma^t C_-(\tau_1,\xi)\,d\tau_1\int_\sigma^{\tau_1} C_-(\tau_2,\xi)\,d\tau_2 + \cdots\right)\boldsymbol{e}^j$$

for all $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^n\backslash 0$. Then we can estimate

$$\left|\boldsymbol{a}_-^j(t,\xi;\sigma)\right| \le e^{\int_\mathbb{R}\|C_-(s,\xi)\|_{(L^\infty(\mathbb{R}^n\backslash 0))^{m^2}}\,ds} \le e^{c\int_\mathbb{R}\|\partial_s\mathscr{N}(s,\xi)\|_{(L^\infty(\mathbb{R}^n\backslash 0))^{m^2}}\,ds}. \tag{5.2}$$

Put

$$\boldsymbol{w}_-^j(t,\xi;\sigma) = \mathscr{N}(t,\xi)^{-1}\Phi(t,\xi)\boldsymbol{a}_-^j(t,\xi;\sigma), \quad j = 0,1,\dots,m-1.$$

Then we see from (5.1) that each $\boldsymbol{w}_-^j(t,\xi;\sigma)$ satisfies the following equation:

$$D_t\boldsymbol{w}_-^j(t,\xi;\sigma) = A(t,\xi)\boldsymbol{w}_-^j(t,\xi;\sigma).$$

It follows from (5.2) that the $\boldsymbol{a}_-^j(t,\xi;\sigma)$ exist uniformly in $\xi \in \mathbb{R}^n\backslash 0$ for all $t \in \mathbb{R}$, and hence, we can estimate

$$\left|D_t\boldsymbol{a}_-^j(t,\xi;\sigma)\right| \le \|C_-(t,\xi)\|\left|\boldsymbol{a}_-^j(t,\xi;\sigma)\right| \le c_1 e^{c\int_\mathbb{R}\|\partial_\tau\mathscr{N}(\tau,\xi)\|_{(L^\infty(\mathbb{R}^n\backslash 0))^{m^2}}\,d\tau}\|\partial_t\mathscr{N}(t,\xi)\|,$$

where we denote by $\|A\|$ a (standard) matrix norm of matrices A. Since $\partial_t\mathscr{N}(\cdot,\xi) \in L^1\left(\mathbb{R};(L^\infty(\mathbb{R}^n\backslash 0))^{m^2}\right)$, these estimates imply that, for $|t| \le |\sigma|$,

$$\begin{aligned}\left|\boldsymbol{a}_-^j(t,\xi;\sigma) - \boldsymbol{e}^j\right| &= \left|\int_\sigma^t \partial_t\boldsymbol{a}_-(\tau,\xi;\sigma)\,d\tau\right| \\ &\le c_1 e^{c\int_\mathbb{R}\|\partial_\tau\mathscr{N}(\tau,\xi)\|_{(L^\infty(\mathbb{R}^n\backslash 0))^{m^2}}\,d\tau}\int_{|t|}^{|\sigma|}\|\partial_s\mathscr{N}(s,\xi)\|_{(L^\infty(\mathbb{R}^n\backslash 0))^{m^2}}\,ds.\end{aligned} \tag{5.3}$$

In particular, the family $\left\{\boldsymbol{a}_-^j(t,\xi;\sigma)\right\}_{\sigma\in\mathbb{R}}$ is uniformly bounded in σ,ξ and equi-continuous on every bounded t-interval. Hence there exists a sequence $\{\sigma_\ell\}_{\ell=1}^\infty$ such that

$$|\sigma_1| < |\sigma_2| < \cdots, \quad |\sigma_\ell| \longrightarrow \infty \quad (\ell \longrightarrow \infty),$$

and the limits

$$\boldsymbol{a}_-^j(t,\xi) = \lim_{\ell\to\infty} \boldsymbol{a}_-^j(t,\xi;\sigma_\ell)$$

exist uniformly in $\xi \in \mathbb{R}^n\backslash 0$ on every bounded t-interval. Moreover, the limits

$$\boldsymbol{w}_-^j(t,\xi) = \lim_{\ell\to\infty} \boldsymbol{w}_-^j(t,\xi;\sigma_\ell) = \mathscr{N}(t,\xi)^{-1}\Phi(t,\xi)\boldsymbol{a}_-^j(t,\xi)$$

also exist. Hence the $\boldsymbol{a}_-^j(t,\xi)$ are solutions of $D_t\boldsymbol{a}_-^j(t,\xi) = C_-(t,\xi)\boldsymbol{a}_-^j(t,\xi)$, and the $\boldsymbol{w}_-^j(t,\xi)$ are solutions of $D_t\boldsymbol{w}_-^j(t,\xi) = A(t,\xi)\boldsymbol{w}_-^j(t,\xi)$. Putting $\sigma = \sigma_\ell$ in (5.3), and letting $\ell \to \infty$, with t fixed, we see that

$$\left|\boldsymbol{a}_-^j(t,\xi) - \boldsymbol{e}^j\right| \le c_1 e^{c\int_{\mathbb{R}} \|\partial_\tau \mathscr{N}(\tau,\xi)\|_{(L^\infty(\mathbb{R}^n\backslash 0))^{m^2}}\,d\tau} \int_{-\infty}^t \|\partial_s\mathscr{N}(s,\xi)\|_{(L^\infty(\mathbb{R}^n\backslash 0))^{m^2}}\,ds \tag{5.4}$$

for all $t \in \mathbb{R}$. The uniqueness of each $\boldsymbol{a}_-^j(t,\xi)$ is obvious.

Now we can write, by putting $\boldsymbol{r}_-^j(t,\xi) = \boldsymbol{a}^j(t,\xi) - \boldsymbol{e}^j$,

$$\boldsymbol{w}_-^j(t,\xi) = \mathscr{N}(t,\xi)^{-1}\Phi(t,\xi)\big(\boldsymbol{e}^j + \boldsymbol{r}_-^j(t,\xi)\big), \tag{5.5}$$

where the $\boldsymbol{r}_-^j(t,\xi)$ satisfy

$$\boldsymbol{r}_-^j(t,\xi) \longrightarrow \mathbf{0} \text{ uniformly in } \xi \in \mathbb{R}^n\backslash 0 \text{ as } t \longrightarrow -\infty \tag{5.6}$$

on account of (5.4). It remains to prove that

$$W_-(t,\xi) \equiv \big(\boldsymbol{w}_-^1(t,\xi),\dots,\boldsymbol{w}_-^{m-1}(t,\xi)\big)$$

is the fundamental matrix for

$$D_t\widehat{V} = A(t,\xi)\widehat{V}.$$

Taking the determinant of $W_-(t,\xi)$, we have, by using (5.6), $\mathscr{N}(t,\xi)^{-1} \to \mathscr{N}_-(\xi)^{-1}$ uniformly in $\xi \in \mathbb{R}^n\backslash 0$ $(t \to -\infty)$ and $|\det\Phi(t,\xi)| = 1$, that

$$|\det W_-(t,\xi)| \longrightarrow |\det\mathscr{N}_-(\xi)|^{-1} \ne 0 \text{ uniformly in } \xi \in \mathbb{R}^n\backslash 0 \text{ as } t \longrightarrow -\infty.$$

Hence there exists a number $t_0 \le 0$ such that

$$\det W_-(t,\xi) \ne 0 \quad \text{for all} \quad t \le t_0.$$

Since $W_-(t,\xi)$ satisfies $D_tW_-(t,\xi) = A(t,\xi)W_-(t,\xi)$, it follows from Abel-Jacobi formula that

$$\det W_-(t,\xi) = \det W_-(t_0,\xi)\exp\int_{t_0}^t \mathrm{tr}A(s,\xi)\,ds \ne 0$$

for all $\xi \in \mathbb{R}^n \setminus 0$ and all $t \in \mathbb{R}$. This means that $W_-(t,\xi)$ is the fundamental matrix for

$$D_t\widehat{U} = A(t,\xi)\widehat{U}.$$

This completes the proof of Proposition 5.1. □

Completion of the proof of Theorem 1.2. Let us find a solution $U(t,x)$ of $D_tU = A(t,D_x)U$ such that

$$\|V_\pm(t,\cdot) - U(t,\cdot)\|_{(L^2(\mathbb{R}^n))^m} \longrightarrow 0 \quad (t \longrightarrow \pm\infty). \tag{5.7}$$

Going back to (3.2)–(3.3) in the proof of Theorem 1.1 (i), we have

$$\begin{aligned} W_\pm(t,\xi) &= \mathscr{N}_\pm(\xi)^{-1}\Phi_\pm(t,\xi)D_\pm(\xi) \\ &\quad + \big(\mathscr{N}(t,\xi)^{-1} - \mathscr{N}_\pm(\xi)^{-1}\big)\Phi_\pm(t,\xi)D_\pm(\xi) + \mathscr{N}(t,\xi)^{-1}\Psi_\pm(t,\xi). \end{aligned}$$

Thus putting

$$\widehat{U}(t,\xi) = \sum_{j=0}^{m-1} W_\pm(t,\xi)D_\pm(\xi)^{-1}\boldsymbol{e}^j \widehat{f}_j^{\pm}(\xi),$$

we can decompose $\widehat{U}(t,\xi)$ into three terms:

$$\begin{aligned} \widehat{U}(t,\xi) =& \widehat{V}_\pm(t,\xi) + \sum_{j=0}^{m-1} \big(\mathscr{N}(t,\xi)^{-1} - \mathscr{N}_\pm(\xi)^{-1}\big)\Phi_\pm(t,\xi)\boldsymbol{e}^j \widehat{f}_j^{\pm}(\xi) \\ &+ \sum_{j=0}^{m-1} \mathscr{N}(t,\xi)^{-1}\Psi_\pm(t,\xi)D_\pm(\xi)^{-1}\boldsymbol{e}^j \widehat{f}_j^{\pm}(\xi). \end{aligned}$$

It can be readily seen that this $\widehat{U}(t,\xi)$ satisfies

$$D_t\widehat{U}(t,\xi) = A(t,\xi)\widehat{U}(t,\xi).$$

Since $\mathscr{N}(t,\xi)^{-1} - \mathscr{N}_\pm(\xi)^{-1} \to O$ uniformly in $\xi \in \mathbb{R}^n\backslash 0$ and $\Psi_\pm(t,\xi) \to O$ for each $\xi \in \mathbb{R}^n\backslash 0$ as $t \to \pm\infty$, we conclude from Plancherel's identity and Lebesgue's dominated convergence theorem that (5.7) is true. Moreover, the mappings

$$\mathscr{W}_\pm : V_\pm(0,\cdot) \longmapsto U(0,\cdot) = \sum_{j=0}^{m-1}\Big[W_\pm(0,\xi)D_\pm(\xi)^{-1}\boldsymbol{e}^j \widehat{f}_j^{\pm}(\xi)\Big](\cdot)$$

are bijective and bounded on $\big(L^2(\mathbb{R}^n)\big)^m$. The proof of Theorem 1.2 is finished. □

Acknowledgements The first author was supported by Grant-in-Aid for Scientific Research (C) (No. 21540198), Japan Society for the Promotion of Science. The second author was supported in parts by the EPSRC grant EP/E062873/1 and EPSRC Leadership Fellowship EP/G007233/1.

The proof of Lemma 4.2 is based on the private communication with Professors Sergio Spagnolo and Marina Ghisi when the first author was visiting the University of Pisa in the sabbatical year. The authors would like to express their sincere gratitude to them. The authors would like to thank also the referees for giving many valuable comments on the first version of the manuscript.

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